X. Figure captions

Figure 1: Coronal (a) and axial (b) views for the source space and sensors used in the simulations. (c)-(f): Sagittal views showing the location of each current dipole test source.

Figure 2: Generalised MFT reconstructions from computer generated data with a single point source. (a) p values: $p = -1$ (linear), (b) $p = 0$ (Standard MFT), (c) $p = 1$ (closest m-FOCUSS equivalent to standard FOCUSS) and (d) a higher value ($p = 3$). The location and direction of the true source is marked by a heavy arrow, and the solution is shown by light arrows, and in grey-scale contour plots of intensity.

Figure 3: Reconstructions from (computer generated) data generated by continuous current vortex confined to level 6. As before, Figures 2a,2b,2c and 2d show respectively generalised MFT solutions obtained with different p values: $p = -1$ (linear), $p = 0$ (Standard MFT), $p = 1$ (closest m-FOCUSS equivalent to standard FOCUSS) and a higher value ($p = 2$).

follows from these equations that it is not natural to have sparse solutions for the current sources unless extra factors of powers of j are introduced in the iteration equations defining the source reconstruction process. There seems to be little natural cause for such extra factors, although that has to be investigated further.

On the other hand it was realised that for $p = 0$, the simplest form of MFT, there is great pressure on the reconstructed source to be sparse. The remaining weight can be chosen (trained) in a variety of ways, by stitching it up with tests with computer generated data, up to the most general one using a neural network, to achieve sensitivity to sparse distributed sources in various regions of the source space.

The more direct fixed point analysis developed in sections 4 to 6 (discussed only briefly because of space limitation) together with the variational approach initiated in section 7 point the way for a system analysis to be conducted in the presence of noise and identify the need for regularisation of the lead field overlap matrix.

A principle was delineated, the Principle of Least Sensitivity, in terms of the sensitivity of the source current strength to variations in the data. The Principle requires that there is independence of the current strength variations to the actual value of the current strength at the point in question. This corresponds to achieving stability of iteration under the effects of noise in the data. It was then shown that only simple MFT satisfied the principle; all other values of the order of the lead field expansion produced current source variations which have enhanced and deleterious sensitivity either to regions of large current strength (for $p < 0$) or to small variations (for $p > 0$). Simulations in this paper, as well as numerous MFT studies with computer generated data supported these statements.

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The arrangement used for the simulations was taken from an actual experiment with auditory stimuli. Figure 1 shows the source space, sensors and some of the test sources used in the simulations. Part 1a and 1b show as coronal and axial view respectively. The MR image is superimposed with the projections of the sensors onto the display plane (each sensor projection is marked by an asterisk). The source space is a hemisphere which best fits the brain hemisphere next to the sensors. For display reasons the continuous solutions within the source space are computed along 9 consecutive sagittal planes. The intersection between a display level and a coronal or an axial slice is a straight line. The intersection between the 9 sagittal display levels and the coronal and axial views are marked in Figures 1a and 1b, beginning with the deepest level (level 1) and ending with the most superficial level (level 9). Figures 1c, 1d, 1e and 1f show as heavy filled triangles the locations of the four test current dipoles, superimposed on the sagittal slice containing the test source; the dipole at level 3 corresponds to generator in the amygdala while the one at level 5 in the general area of the hippocampus and insula. The more superficial dipoles, at levels 7 and 9, correspond to generators at deep (1e) and very superficial parts of the auditory cortex respectively. In addition to the dipole position, the source space circular outline at each level is shown, together with the projection of the centre of the conducting sphere on that level.

Figure 2 shows reconstructions with the test current dipoles. Sub-figures display reconstructions for the linear case (p=-1, Figure 2a), simple MFT (p=0, Figure 2b) and for generalised MFT with p=1 (standard m=FOCUSS, Figure 2c) and p=3 (Figure 2d). In each sub-figure the reconstruction for each test dipole is shown on a different column, with the separate displays for levels 3,5,7 and 9. The circular source space boundary is shown at each level, and the skull and brain outline for only the top two levels. The current dipole is marked by a heavy arrow at the correct level, while the continuous current density solution is shown as a grey-scale intensity plot and as an arrow map (with cut-off at 0.45 of the maximum arrow computed separately at each column).

Figure 3 shows reconstructions for a continuous 2D source. We have used a vortex source confined to level 6. The magnetic field generated from this source was calculated from the sum of current dipoles with strength defined by the local value of the continuous distribution and the value of the Gaussian quadrature weight so that the sum corresponds to the Gaussian quadrature integral over the active source space plane. As before sub-figures display reconstructions for the linear case (p=-1, Figure 3a), simple MFT (p=0, Figure 3b) and (p=1,Figure 3c) and (p=3, Figure 3d). The same conventions have been used as for Figure 2, except that the target distribution is not displayed, because its continuous nature would have cluttered the display at the active level. Its form is well approximated by the continuous solution in Fig 3a.

In all forward and inverse computations we have used the conducting sphere model, and we have used 49 integration points (Gaussian quadrature) for the computation of the 2D integral of the flux through the magnetometer coils, and 1000 integration points (Gaussian quadrature) for the overlap integrals though the source space. We have used the standard MFT procedures, as they have repeatedly used and described in the past, for all source configurations and p values: a one iteration step was used for the training session and the analysis of the computer generated data. The same w_0 weight (determined by the training session) was used throughout. Any other choice makes the comparison difficult: if more iterations are carried out a different choice for $w_0(\mathbf{r})$ will be required for each iteration number and p value, and even relative source strengths in the source space for p values other than -1 and 0. The linear case is independent of source strength and it is therefore unaffected by iteration, while simple MFT changes appreciably as a result of the first (and only) iteration, but little more if more iterations are used. In contrast, for p=1 and especially p=3 each iteration produces a big change emphasising more the strongest surviving source of the previous iteration step.

Figures 2 and 3 show that in general the solutions for different p-values become sharper as p increases from the weighted minimum norm case ($p = -1$) to MFT ($p = 0$), through to higher p-values. The higher p values describe better a single point like source, as shown in figures 2c and 2d. The focal nature of the solutions becomes a big disadvantage when sources of different strength or distributed sources are present: the weaker sources are drastically reduced, and quickly eliminated as the iterations proceed, as demonstrated in Figure 3 and in earlier simulations [17]. The solutions for the weighted minimum norm case always produce extended distributions; as figure 3 shows weighted minimum norm and standard MFT can describe distributed activity better than the higher p values. In the example in Figure 3 where the true vortex like source is confined to a single plane the weighted minimum norm describes better the in-plane source profile, while MFT better describes the confinement to one plane.

IX. Summary and Conclusions

In this paper we have discussed the lead field approach to the bioelectromagnetic inverse problem. We have emphasised the MEG case, where we have explicitly used the vectorial and analytic character of the lead fields themselves. We have considered in some detail the general lead field expansion for the unit vector source current $\hat{\mathbf{j}}$, in terms of the powers, p, of $|\mathbf{j}|$; the expansion was defined as order p where p is the highest power allowed. If a single power is used, then the standard MFT case has $p = 0$, and the linear expansion has $p = -1$. The closest lead field equivalent to standard FOCUSS (m-FOCUSS) has p=1.

It was shown that the generalised lead field expansions, for $p \neq 0$, it is possible to reduce the reconstruction problem to a non-linear fixed point problem for matrices of size $N \times N$ (where N is the number of measurement coils). It expansion of order p given by (18). That can be done by variational analysis, leading to the first order variational equation of the non-linear equation (32) for the variations $\delta\mathbf{j}(\mathbf{r})$ in terms of the variations $\delta\mathbf{d}$ in the measurements (or in other parameters, such as the lead field parameters if so desired):

$$\begin{gathered}(p+1)\int(\delta\mathbf{j}(\mathbf{s})\cdot\mathbf{j}(\mathbf{s}))\,j^{p-1}(\mathbf{s})w_p(\mathbf{s})\\ \times\left(\mathbf{d}^T\mathbf{L}^{-1}(j^{p+1})\mathbf{M}(\mathbf{s})\mathbf{L}^{-1}(j^{p+1})\mathbf{M}(\mathbf{r})\mathbf{L}^{-1}(j^{p+1})\mathbf{d}\right)ds\\ -p\,(\delta\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}))\,j^{-2p-2}(\mathbf{r})\\ =\left(\delta\mathbf{d}^T\mathbf{L}^{-1}(j^{p+1})\mathbf{M}(\mathbf{r})\mathbf{L}^{-1}(j^{p+1})\mathbf{d}\right)\end{gathered} \tag{39}$$

It is also possible to consider the case of small variations about the true source current density satisfying (32) with no change in the measurement values $\mathbf{d}$, so corresponding, for example, to the change of $\mathbf{j}$ under an iteration of equation (32).

In both cases a sum rule for $\delta\mathbf{j}(\mathbf{r})$ is obtained, for $p \neq -1$, of the form

$$\begin{gathered}(p+1)\int(\delta\mathbf{j}(\mathbf{s})\cdot\mathbf{j}(\mathbf{s}))\,j^{p-1}(\mathbf{s})w_p(\mathbf{s})d\mathbf{s}\cdot\\ \left(\mathbf{d}^T\mathbf{L}^{-1}(j^{p+1})\mathbf{M}(\mathbf{s})\mathbf{L}^{-1}(j^{p+1})\mathbf{M}(\mathbf{r})\mathbf{L}^{-1}(j^{p+1})\mathbf{d}\right)j^{2p}(\mathbf{r})\\ -p\,(\delta\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}))\,j^{-2p-2}(\mathbf{r})=\Lambda(\mathbf{r},\mathbf{d},j)\end{gathered} \tag{40}$$

where $\Lambda(\mathbf{r},\mathbf{d},j)$ is a given function of its variables and depends less strongly on j than does the left hand side (and is zero in the case of iteration).

We will now show from (40) that it is only for $p = 0$, the case of MFT, that there is least sensitivity to modification of the effect of changes in $\delta\mathbf{j}(\mathbf{r})$ due to the factor $j^p(\mathbf{r})$ or p in the evaluation of the contributions to the left hand side of the sum rule at $\mathbf{r}$ due to changes $\delta\mathbf{d}$ in the data or from iteration (as in (39)). There are two terms on the left hand side of (39) to be investigated.

The first of these involves an integral over all source position of the variation in the source current. The crucial factor in the integrand is the factor of order j^p . In the case of $p > 0$ there is enhanced sensitivity to large values of j arising from this factor, so the solutions can become unstable due to these regions of source space. Small errors in source reconstruction will become amplified. There is a corresponding reduction in sensitivity from this factor in regions of small j. In the case that $p < 0$ (excepting the linear case $p = -1$) equation (39) indicates that there is enhanced sensitivity to variations in the source data from regions of small j.

The second term on the left hand side of (39) vanishes for the MFT case $p = 0$. For other values of p there is again modification of the variation in source strength by a factor equal to the inverse of that quantity raised to the power $(-2p-1)$. These effects are in the opposite region of source strengths to the first term in (39). This second term also has deleterious effects on the convergence of the iteration process or on dependence on noise in the data differentially across different regions of source space unless it is totally cancelled. That requires $p = 0$.

It is possible to deduce from (39) that if it is desired to have a lead field expansion which has a sensitivity to variations of the measurement data which avoids enhancement of errors in the source strength values (modulo the global effect of j due to the dependence of the right hand side of (39)), or under iteration leads to no similar enhancement, then the value $p = 0$ is singled out. This is MFT. We thus conclude that MFT satisfies the principle of least sensitivity: the solutions are least sensitive both to variations of the data and also to iterations of the nonlinear norm constraints (32)).

From our above analysis, standard MFT, with $p = 0$, is the unique solution to this principle amongst the set of lead field expansions of arbitrary order p. However we have had to exclude the linear case ($p = -1$) from our discussions. In that case a direct analysis can be made of the dependence of $\delta\mathbf{d}$ on $\delta\mathbf{j}(\mathbf{r})$; the result follows from the above equations, and is

$$\delta\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r})=\left(\delta\mathbf{d}^T\ \mathbf{L}^{-1}(1)\mathbf{M}(\mathbf{r})\mathbf{L}^{-1}(\mathbf{1})\mathbf{d}\right) \tag{41}$$

so that the inverse power of $\mathbf{j}(\mathbf{r})$ entering (41) is present, as expected from extending (33) to $p = -1$ naively, and extracting the value of $\delta\mathbf{j}(\mathbf{r})$. This extension is seen to be correct since equation (32) reduces to that for the linear case when $p = -1$. Thus the linear case does not have least sensitivity to changes in j, leaving only MFT with this property from the whole set of lead field expansions of any order.

It is possible to consider the energy, either at a point as $j^2(\mathbf{r})$ or its integral over the whole head. Both of these quantities depend least sensitively on measurement variations in the case of the linear expansion, $p = -1$. However they are not the primary quantities of relevance to solving the inverse problem. If the quantity $\mathbf{j}(\mathbf{r})$ is accepted as the primary one, then the advantage of the linear approach is lost , and the MFT solution comes to the fore.

VIII. Simulations

The simulations described in this section are not comprehensive. They are chosen to bridge the gap between the many earlier simulation studies performed together with actual MFT analysis of real data where only one side of the head was covered with (usually 37) sensors, and more recent studies where helmet like probes allow the coverage of the whole head with well over 100 sensors. For the simulations we have used 49 sensors covering a fairly large part of one hemisphere. For the computations the conducting sphere model was used with sphere centre taken from a best fit of the inner part of the skull below the sensor array (which for this case was 1.5 cm on the opposite hemisphere to the side covered by sensors). A rather large source space was used to test the limits of what can be achieved, and to provide justification through examples for the various statements and theorems developed so far.

VI. Training and Sparse Solutions of MFT

It is important to note that the iteration step in generalised MFT introduces an new element. Compared to directly analysing the equations (33), these are rewritten by inclusion of extra factors of the source strength, as in (12) and (13) to lead to

$$j^2(\mathbf{r}) = \left(\mathbf{d}^T\mathbf{L}^{-1}(|\mathbf{j}|^{p+1})\mathbf{M}(\mathbf{r})\mathbf{L}^{-1}(|\mathbf{j}|^{p+1})\mathbf{d}\right) j^{2p+2}(\mathbf{r}) \quad (34)$$

The iteration of (34) will lead to the same solutions as (33) except for the set of zeros of j. However it is to be expected for general positions of the coils and data values d that the term in brackets on the right hand side of (33) will not be zero anywhere. Thus it would seem that there is no clear justification to set j to be zero at a set of points except by further knowledge brought in from outside the present framework.

There is a further feature to note, however, in the relation between the weighted minimum norm lead field expansion (18) and that arising from the weighted minimum norm optimisation process of lemma 4.1. For the solution of (25) is only identical to the weighted minimum norm expression (18) at non-zero values of j. Outside the support of j the expression (24) is not defined for positive n, and was then set to zero as part of the FOCUSS algorithm. However there is no problem for the right hand side of (18) at zeros of j, unless p is negative (where p and n are identified through lemma 4.1). Thus the domain of definition of the lead field expansion are dependent on exactly what they are expansions of, either of $\hat{\mathbf{j}}(\mathbf{r})$ (as in generalised MFT through (18)) or of $\hat{\mathbf{j}}(\mathbf{r}) \cdot |\mathbf{j}(\mathbf{r})|^{-p}$ as in lemma 4.1. This ambiguity makes the detailed comparison of MFT with FOCUSS more problematical, and simulations of the two algorithms would be valuable at this point. That will be discussed elsewhere.

No such ambiguity in the definition of the object being expanded in the lead fields exists in the case of $p = 0$. For that, the simple MFT of Ioannides [17], [15] uses a two step approach coupling a training process for choosing the weight $w_0(\mathbf{r})$ to the equivalent iteration to (34). In order to understand this process better the case of (34) at $p = 0$ will now be considered in more detail. The iteration process so defined becomes the construction of the fixed points of the non-linear system (from (34) with re-insertion of the source independent weight $w_0(\mathbf{r})$):

$$j^2(\mathbf{r}) = w_0^2(\mathbf{r})\left(\mathbf{d}^T\mathbf{L}^{-1}(|\mathbf{j}|)\tilde{\mathbf{M}}(\mathbf{r})\mathbf{L}^{-1}(|\mathbf{j}|)\mathbf{d}\right) j^2(\mathbf{r}) \quad (35)$$

with,

$$\tilde{\mathbf{M}}(\mathbf{r}) = w_0^{-2}(\mathbf{r})\mathbf{M}(\mathbf{r}) = (\phi_m(\mathbf{r}) \cdot \phi_n(\mathbf{r})) \quad (36)$$

We may deduce from the simple result that iteration of (35) leads to the set of source values at each point which are either zero or satisfy (28). Thus it is possible to obtain sparse solutions to the reconstruction problem (35), since there may only be a restricted set of points at which it is possible to satisfy (28), so have $j \neq 0$. At all other points j must vanish. At the points where $j \neq 0$ it is possible to choose the weight w_0 so as to satisfy (3) as

$$w_0^2(\mathbf{r}) = \left(\mathbf{d}^T\mathbf{L}^{-1}(|\mathbf{j}|)\tilde{\mathbf{M}}(\mathbf{r})\mathbf{L}^{-1}(|\mathbf{j}|)\mathbf{d}\right)^{-1} \quad (37)$$

(where it is assumed that the denominator on the right hand side of (37) is not zero, as is to be expected for general dispositions of the measurement coils). However if equation (37) is to be used to estimate $w_0(\mathbf{r})$ some choice must be made for the sources $\mathbf{j}$ which enter into its right hand side.

The technique used so far, and explicitly described in [14], [15] to answer this is to take a set of 'training data' as a set of source currents $\{\mathbf{j}^{(\alpha)}(\mathbf{r})\}$, where the parameter runs over labels for the training set, $\alpha = 1, 2, ..., T$. These sources are assumed to be non-zero in separate regions of the source space. As such, the relevant region over which the weight function w_0 is specified is therefore restricted, in the case of source $\mathbf{j}^{(\alpha)}(\mathbf{r})$, only to the support of this source. The weight function is then 'stitched' together. In all published MFT applications, so far, this was achieved by assumption of a very simple form for the weight, a Gaussian, with centre on the centre of the head. On a few recent cases where a large source space was used with comparatively incomplete sensor coverage, a probability weight with two Gaussians was used. The width(s) of the one (or two) Gaussian(s) was adjusted so as to give maximal source space reconstruction for the training sources at their various depths.

Whilst such an approach seems satisfactory at a certain level it is possible to extend it to a larger class of unknown sources if a more general weight w_0, such as one provided by a one-hidden layered neural network through the 'Universal Approximation Theorem' of neural network theory. In that case a possible error term with which to train the weights in such a network would be the mean squared error of the reconstructed current:

$$\sum_\alpha \int \left|\mathbf{j}^{(\alpha)}(\mathbf{r}) - \mathbf{j}^{(\alpha),(rec)}(\mathbf{r})\right|^2 d\mathbf{r} \quad (38)$$

where the superscript (rec) denotes reconstructed. Investigations are now in progress to determine if such a training scheme would lead to superior results than those already available.

VII. Sensitivity Analysis

We will now attempt to determine a value of p to achieve optimal sensitivity to changes in the data. That has already been done in terms of analysis of the variances of measured fluctuations in the measured currents by Hämäläinen et al [13] for the linear case. Let us extend this sensitivity analysis to the case of the lead field

The constraint applied to (23) is that of the measured values, equation (1) above. We note that minimisation of these cost functions corresponds to the minimum L_{-n} norm solution to the constraint equations. Let us state and prove an important lemma on the relation of m-FOCUSS to the expansion (18).

Lemma 4.1
For $p = n + 1$, the solutions to the optimisation problem (23) with the constraints (1) have the form (18).

Proof.
The measurement constraints may be implemented in the standard manner by Lagrange multipliers λ_m, as before, to give the unconstrained action

$$\Omega = \int |\mathbf{j}(\mathbf{r})|^{-n}\, d\mathbf{r} - \sum_m \lambda_m \left(d_m - \int \phi_m(\mathbf{r}).\mathbf{j}(\mathbf{r}) d\mathbf{r} \right) \quad (24)$$

(together with a similar one for the case $n = 0$ in (24) in which the power of $|\mathbf{j}(\mathbf{r})|$ in the first integral is replaced by the logarithm of $|\mathbf{j}(\mathbf{r})|$). It is immediate that variation of Ω with respect to the Lagrange multipliers λ_m leads to the measurement constraints (1), as it should. However it is the variation of Ω with respect to the current source distribution which leads to the desired result. For the variation has value

$$\frac{\delta\Omega}{\delta\mathbf{j}(\mathbf{r})} = \sum_m \lambda_m \phi_m(\mathbf{r}) - n\hat{\mathbf{j}}(\mathbf{r})\, |\mathbf{j}(\mathbf{r})|^{-(n+1)} \quad (25)$$

Setting (25) to zero leads to the desired result, with $p = n + 1$. A similar result is seen for the case of $n = 0$ in (24), leading to the $p = 1$ expansion of equation (18).

Thus although m-FOCUSS appears to be a non-parametric approach to solving the inverse problem in the presence of the constraints (1), it leads to solutions which are expressible in terms of the weighted minimum norm lead field expansions of definition 5.1 and equation (18). Moreover the lemma proves that m-FOCUSS is nothing but generalised MFT, however iterated using different methods (quasi Newton) and different criteria on when a solution is non-zero or is to be set to zero. From now on in this and the next section we will only therefore consider the $|\mathbf{j}(\mathbf{r})|^p$ weighted expansions, and hence the generalised MFT. We will return to m-FOCUSS in a discussion of the simplest MFT, with order $p = 0$.

V. Analysis of MFT

The argument for MFT is still incomplete, since its structure has not yet been properly analysed. A first step towards doing that is to obtain the associated norms of the current densities. For the expressions (2) and (3) one has:

$$\int \frac{|\mathbf{j}(\mathbf{r})|^2}{w_0(\mathbf{r})} d\mathbf{r} = \left(\mathbf{d}^T \mathbf{L}^{-1}(1)\mathbf{d}\right) \quad (26)$$

whilst for the MFT solution (7) and (11) discussed above it may be shown that

$$\int \frac{|\mathbf{j}(\mathbf{r})|}{w_0(\mathbf{r})} d\mathbf{r} = \left(\mathbf{d}^T \mathbf{L}^{-1}(|\mathbf{j}|)\mathbf{d}\right) \quad (27)$$

Note the difference in the powers of the two norms (26) and (27), which arise naturally from the nature of the solutions for the sources in the two cases. For the simplest form of MFT there is no simple form for the squared norm of the source current. Indeed it is only possible to determine the squared norm for the MFT solution after the local source strength $|\mathbf{j}(\mathbf{r})|$ has been calculated from the non-linear MFT equations (11). Thus MFT does not lead to the minimum-squared-norm solution. What, then, is the basic assumption behind the simplest form of MFT, which may single it out from other possible choices of the weight function $w(\mathbf{r}, |\mathbf{j}(\mathbf{r})|)$? It is possible to rewrite (11) more transparently as

$$\left(\mathbf{d}^T \mathbf{L}^{-1}(|\mathbf{j}|)\mathbf{M}(\mathbf{r})\mathbf{L}^{-1}(|\mathbf{j}|)\mathbf{d}\right) = 1 \quad (28)$$

(from which (27) may be immediately obtained) at any source point $\mathbf{r}$, where the matrix $\mathbf{M}(\mathbf{r})$ is formed from scalar product of the the pairs of lead fields at the point $\mathbf{r}$:

$$\mathbf{M}(\mathbf{r}) = (\phi_m(\mathbf{r}) \cdot \phi_n(\mathbf{r}))w_0^2(\mathbf{r}) \quad (29)$$

We will now proceed to develop the same approach to the solution of the measurement constraints (1) in the case of the weighted norm expansion (24), for a particular but unspecified value of the order p (but not zero). The solution of the data constraints (1), similar to the solutions (14) for $p = -1$ and (14) for $p = 0$, using (18), is now

$$\mathbf{A} = \left(\mathbf{L}(|\mathbf{j}|^{p+1}\right)^{-1}\mathbf{d} \quad (30)$$

where

$$L_{mn}(|\mathbf{j}|^{p+1}) = \int (\phi_m(\mathbf{r}) \cdot \phi_n(\mathbf{r}))\, |\mathbf{j}(\mathbf{r})|^{p+1}\, w_p(\mathbf{r}) d\mathbf{r} \quad (31)$$

and (28) now becomes. for the expansion of order p,

$$1 = j^{2p}(\mathbf{r}) \left[\mathbf{d}^T \mathbf{L}^{-1}(|\mathbf{j}|^{p+1})\mathbf{M}(\mathbf{r})\mathbf{L}^{-1}(|\mathbf{j}|^{p+1})\mathbf{d}\right] \quad (32)$$

It is appropriate to attempt to analyse the system of equations (32) from the mathematically powerful fixed point aspect, instead of subjecting them only to the rigors of a less precisely defined simulation campaign. To do this we manipulate equation (32), to give, for $p \neq 0$,

$$j(\mathbf{r}) = \left[\mathbf{d}^T \mathbf{L}(|\mathbf{j}|^{p+1})\mathbf{M}(\mathbf{r})\mathbf{L}^{-1}(|\mathbf{j}|^{p+1})\mathbf{d})\right]^{-\frac{1}{2p}} \quad (33)$$

For $p \neq 0$, (33) can be used to reduce the infinite system of equations (32) to an NxN set of equations, where N is the number of sensors. Fixed point methods then allow the deduction of existence and uniqueness results but they do not show how stable the iteration will be. We will return to the question of stability after we complete the earlier discussion on training and sparsity of solutions.

represents only the direction of the currents. Their strength must be determined more explicitly from the data condition (1). That is the simplest form of MFT given by (7), as developed in the non-linear equation (11). However there is still something missing in the argument for (7) versus (3), since each appears to give a maximal reconstruction of the currents. What is the hidden assumption which singles out (7) to be effective under simulation rather than (3)?

The assumption behind expansion (7) appears be based on the way it allows one to control for the bias of solutions, and the prevention of the sources gravitating to the surface of the head closest to the detectors, through the choice of the weight w_0 as a Gaussian centered at the centre of the head. This control can only be exercised with an appropriate choice of the a priori probability weight $w_0(\mathbf{r})$. With the weighted minimum norm this can be done, for example, with equation (16), which has the full current density vector on its left. This can therefore only be achieved with a $w_0(\mathbf{r})$ which caters for both the strength and the direction of the current at each source point $\mathbf{r}_0$. In contrast the simplest form of MFT requires a choice of $w_0(\mathbf{r})$ which is independent of the modulus of the current, but only depends on its location; this is exactly what the training achieves. It is for this reason that MFT is able to cope with just a simple Gaussian form factor for $w_0(\mathbf{r})$. It is only through equation (11), which determines the source strengths, that the data and the weight function operate to allow for a more specific localisation of the sources as specified by the collection of lead fields. It is precisely the process of 'sharpening up' the source distribution [14] at later stages of the iteration of (11), that the localisation is achieved. Such localisation cannot occur through (3), where the source current strengths is not so strongly controlled by the data.

IV. Costs Functions

Before emphasising the special character of MFT we widen the framework. We first introduce the more general expansion (5); this allows us to incorporate a modified FOCUSS (m-FOCUSS) as a natural part of the structure.

We first define the lead field expansion to have order p if the weight w' of equation (5) is proportional to the p'th power of $|\mathbf{j}(\mathbf{r})|$.

The weighted minimum norm of order p has the current source distribution unit vector expanded as the set of lead fields at the point, weighted by the p^{th} power of the current source:

$$\hat{\mathbf{j}}(\mathbf{r}) = \sum_m A_m \phi_m(\mathbf{r}) w_p(\mathbf{r})\, |\mathbf{j}(\mathbf{r})|^p \tag{18}$$

where the source-independent weight $w_p(\mathbf{r})$ is still undetermined.

A. On the properties of Cost functions

A word must be said at this point about the dependence of a cost function on the source vector $\mathbf{j}(\mathbf{r})$. Consider the cost function, as an integrated density, with no coupling between the values of j at different positions:

$$C(j) = \int c(j(\mathbf{r}))d\mathbf{r} + \sum_m \lambda_m \left(\int \phi_m(\mathbf{r})\cdot\mathbf{j}(\mathbf{r})d\mathbf{r} - d_m \right) \tag{19}$$

where the λ_m are Lagrange multipliers for the measurement conditions.

It can be seen that the value of $\frac{\delta c}{\delta \mathbf{j}(\mathbf{r})}$ must rotate as a vector under three-dimensional rotation of all lead fields. For the equations of the source currents arising from (19) we get,

$$\frac{\delta c}{\delta \mathbf{j}(\mathbf{r})} = \sum_m \lambda_m \phi_m(\mathbf{r}) \tag{20}$$

and the right-hand side of (20) rotates as a vector as can be explicitly derived from the well known expressions relating the source and magnetic field in the conducting sphere [26]

It therefore follows that c can only depend on the Euclidean length of j, i.e., $j = \left(j_x^2 + j_y^2 + j_z^2\right)^{\frac{1}{2}}$ and,

$$\frac{\delta c}{\delta \mathbf{j}(\mathbf{r})} = c'(j) \cdot \hat{\mathbf{j}}(\mathbf{r}) \tag{21}$$

The RHS of (21) rotates as a vector as it should.

The above criterion disqualifies FOCUSS at the first fence, since it uses the 'city block' distance,

$$d(\mathbf{j}) = |j_x| + |j_y| + |j_z| \tag{22}$$

and equation (20), with c replaced by d can not be consistent in all reference frames, because it does not possess rotational invariance.

There may exists preferred current directions in the cortex; these could also enter the cost function c in (19). However, $\frac{\delta c}{\delta \mathbf{j}(\mathbf{r})}$ should still rotate as a vector under a rotation of the co-ordinates since such vectors will do also. Thus the above argument against (22) still stands.

In order to be able to proceed with some kind of meaningful comparison we define as the previously introduced m-FOCUSS to be the analogue term in generalised MFT for which the city block distance of FOCUSS is replaced by the Euclidean distance.

B. MFT and (m-)FOCUSS

It can be shown that once the 'city block' distance is replaced by the Euclidean distance then the arguments used in FOCUSS can be redrawn, within the generalised MFT framework which for ease of reference we will still refer to also as m-FOCUSS. This leads to the expansion (18), with the closest equivalent to standard FOCUSS corresponding to $p = 1$. We sketch the analysis below, beginning with the constrained optimisation cost function at the basis of m-FOCUSS,

$$\int ln\,|\mathbf{j}(\mathbf{r})|\, d\mathbf{r} \quad or \quad \int |\mathbf{j}(\mathbf{r})|^{-n}\, d\mathbf{r} \quad (for\ n \neq 0) \tag{23}$$

We see how the expressions (8) and (9), on use of (11), have the correct number of degrees of freedom to allow for a unique reconstruction of the whole source distribution $\mathbf{j}$, not in the null space, both in direction and strength. Moreover it is the non-linearity of equation (11) in the source strength (containing also details of the over-all normalisation of the source strength) which gives power to the MFT approach using equation (7). It is that equation which was solved by iteration in all MFT applications as explicitly described in the two MFT reviews [14], [15]. This equation will be discussed in more detail later; we will describe here briefly the MFT algorithm arising from the conditions (11).

We first multiply both sides of (11) by $j^2(\mathbf{r}_0)$, to give

$$j^2(\mathbf{r}_0) = \left| \sum_m \left((\mathbf{L}(|\mathbf{j}|))^{-1}\, \mathbf{d} \right)_m \phi_m(\mathbf{r}_0) w(\mathbf{r}_0) \right|^2 j^2(\mathbf{r}_0) \quad (12)$$

The algorithm itself is then obtained by solving (12) by iteration for each $\mathbf{r}_0$:

$$j^2_{n+1}(\mathbf{r}_0) = \left| \sum_m \left((\mathbf{L}(|\mathbf{j}_n|))^{-1}\, \mathbf{d} \right)_m \phi_m(\mathbf{r}_0) w(\mathbf{r}_0) \right|^2 j^2_n(\mathbf{r}_0) \quad (13)$$

for $n = 1, 2, \ldots$. This iteration procedure converges to the solution of (12). The resulting solution is expected to be sparse, in that $j(\mathbf{r}_0)$ must vanish at those points $\mathbf{r}_0$ where equation (11) is not satisfied. Those points where equation (11) is satisfied (to within some small error criterion) will have non-zero values of j, which are linked together by the condition (12) involving the effects of sources at different points. This is where the choice of a suitable $w(\mathbf{r}_0)$ is important. We note that in practice for $n = 0$ a uniform $|j(\mathbf{r}_0)|$, i.e. $j_0(\mathbf{r}_0) = 1$ works well. The procedure adopted so far is to 'train' w by choosing it so as to give correct reconstructions for a set of localised sources. This is discussed in more detail later, but we should add here for completeness, that a suitably chosen form of w should provide the optimal level of sparsity of the reconstructed source currents. We will discuss this later when p-values other than 0 (for standard MFT) are considered, but note that a poor choice of w will make (11) more difficult to satisfy. This will lead to a more sparse solution than is appropriate given the measurement values. Thus the choice of p and w are all relevant to the level of sparsity.

If the minimum norm solution (3), had been used (with no j-dependence of w) then (7) would have the right hand side replaced by the total value $\mathbf{j}$ at the source point. The corresponding solution to equation (9) is now the well-known value

$$\mathbf{A} = \mathbf{L}^{-1}(1)\mathbf{d} \quad (14)$$

with no further restrictions on the parameters. This is why we termed this case linear in the introduction, since the whole of the solution (14) has been obtained as a solution to a set of linear equations, in which there is only linear dependence of the current solution on the data.

These two ansatzs for (2), (3) and for (7) are seen to lead to identical solutions in the case of a single dipole at the point $\mathbf{r}_0$, or whenever the current $\mathbf{j}(\mathbf{r}_0)$ has a fixed modulus of either zero or a constant value. In that case the dependence of $\mathbf{L}(|\mathbf{j}|)$ in equation (9) factors out as the term $|\mathbf{j}(\mathbf{r}_0)|$, so that

$$\mathbf{L}(|\mathbf{j}|) = |\mathbf{j}(\mathbf{r}_0)|\, \mathbf{L}(1) \quad (15)$$

It can then easily be seen that the two solutions become identical for all points where $|\mathbf{j}(\mathbf{r}_0)|$ is not zero, with

$$\mathbf{j}(\mathbf{r}_0) = \sum_m \left(\mathbf{L}^{-1}(1)\mathbf{d} \right)_m \phi_m(\mathbf{r}_0) w(\mathbf{r}_0) \quad (16)$$

following from (7) and (15). However identity does not occur for two or more dipoles, of different strength, at different positions and even less is it possible for a continuous distribution of current, if more than one non zero values for $|\mathbf{j}(\mathbf{r}_0)|$ are allowed.

It is possible to develop an analysis of the manner in which MFT works for localised dipoles or localised regions of currents distant from each other. For then $\mathbf{L}(|\mathbf{j}|)$ breaks into block matrices in the subspaces of the lead fields closest to the relevant dipole or current distributions, using the dependence of the lead field vectors $\mathbf{L}_m$ as $\frac{1}{|\mathbf{r}-\mathbf{r}_0|^2}$. In each of these subspaces the single dipole analysis can then be shown to result in the condition on the local source direction $\langle \mathbf{j}(\mathbf{r}_1) \rangle$ at the maximal source point $\mathbf{r}_1$

$$\begin{aligned} \textstyle\sum_{m,n} \left(\phi_m(\mathbf{r}_1) \cdot \left\langle \hat{\mathbf{j}}(\mathbf{r}_1) \right\rangle \right) \left[\phi_m(\mathbf{r}_1) \cdot \phi_n(\mathbf{r}_1) \right]^{-1} \\ \times \left(\phi_n(\mathbf{r}_1) \cdot \left\langle \hat{\mathbf{j}}(\mathbf{r}_1) \right\rangle \right) = 1 \end{aligned} \quad (17)$$

where the summation over m and n is over indices for lead fields that are large in the local block being considered.

So far the detailed analysis leading to (9) and (11) or to (14) in the two cases of starting from equations (7) (with no j-dependence in w in the expansion of the unit current in lead fields) or (2) and(3) (starting with no j-dependence of w in the expansion of the total current) , have not lead to any contradiction. They both lead to a full reconstruction of the source distribution (modulo radial silent currents). So how can they be decided between? Both lead to solutions fixing the other available degrees of freedom. The clinching argument is now apparent from the nature of (11) (for the simplest form of MFT based on (7)) versus (14) (based on (2)) with the full strength of the current distribution at each point in the latter case given by the expansion (2). The argument is as follows. It is expansion (2), with the further assumption (3) which appears to be too strong for accurate reconstruction of distributed currents. It presupposes that it is possible to express both direction *and strength* of current source distributions in one single linear expansion in the lead fields, as (3) requires. The lesser requirement is that such a linear expansion, as given by (7), The notation underlines that the integration is over the source space ß and that not all of $\mathbf{j}(\mathbf{r})$ influences the measurements: the suffix 0 on $\mathbf{j}$ denotes 'observable part', i.e. the part of $\mathbf{j}$ which has a non-zero overlap with the lead fields. The non-observable current source distribution, $\mathbf{j}(\mathbf{r}) - \mathbf{j_0}(\mathbf{r})$ is termed silent current and it belongs to the null space of the measurements. Since we will only consider the observable part, we drop the zero suffix, always remembering that $\mathbf{j}$ refers from now on to the observable part of the source current density. We also drop the subscript ß from the integrals, but in all integrals from now on the integration will always be over the source space ß.

The algorithms proposed so far assume (explicitly or implicitly) that there exists a probability weight $w(\mathbf{r}, \mathbf{j}(\mathbf{r}))$, in general depending on both the source position and on the source vector $\mathbf{j}(\mathbf{r})$ at the source point $\mathbf{r}$, in terms of which it is possible to expand the observable part of the source distribution as

$$\mathbf{j}(\mathbf{r}) = \sum_m A_m \phi_m(\mathbf{r}) w(\mathbf{r}, \mathbf{j}(\mathbf{r})) \tag{2}$$

FOCUSS does not seem to use equation (2) explicitly, but it will be shown later that indeed it does do so. The reason that there is a problem about (2) (modulo the non-uniqueness of the solution to the inverse problem that follows the existence of silent currents) is that there is a great degree of freedom in the choice of $w(\mathbf{r}, \mathbf{j}(\mathbf{r}))$. The simplest choice is that the weight function w be independent of $\mathbf{j}$:

$$w(\mathbf{r}, \mathbf{j}(\mathbf{r})) = w(\mathbf{r}) \tag{3}$$

This gives rise to the well-known minimum norm solution, termed so because it produces the solution which minimises the length, j, of $\mathbf{j}$ at each point $\mathbf{r}$ [13]:

$$j^2(\mathbf{r}) = |\mathbf{j}(\mathbf{r})|^2 = \mathbf{j}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}) \tag{4}$$

The minimum norm approach could be called a 'linear model', since it presents a linear problem to solve for the expansion coefficients A_m in terms of the data, and hence is a linear problem for the source current $\mathbf{j}$. This will be considered in detail later.

Another set of lead field expansions is reached by replacing the left hand side of (2) by the unit vector for the source distribution:

$$\hat{\mathbf{j}}(\mathbf{r}) = \frac{\mathbf{j}(\mathbf{r})}{|\mathbf{j}(\mathbf{r})|} = \sum_m A_m \phi_m(\mathbf{r}) w'(\mathbf{r}, \mathbf{j}(\mathbf{r})) \tag{5}$$

The revised form for the source expansion in lead fields, equation (5), now has a simplest version in which the weight w' is taken to be independent of j, so is of the form

$$w'(\mathbf{r}, \mathbf{j}(\mathbf{r})) = w_0(\mathbf{r}) \tag{6}$$

so giving

$$\hat{\mathbf{j}}(\mathbf{r}) = \sum_m A_m \phi_m(\mathbf{r}) w_0(\mathbf{r}) \tag{7}$$

We note that expression (2) is very broad, with the possibility that $w(\mathbf{r}, \mathbf{j}(\mathbf{r}))$ can be expressed as a power series in $|\mathbf{j}(\mathbf{r})|$. It is evident that equation (7) is already of the form (2), with $w(\mathbf{r}, \mathbf{j}(\mathbf{r})) = w_0(\mathbf{r}) \, |\mathbf{j}(\mathbf{r})|$, The form (7) is the basis of the simplest form of MFT. It was selected amongst a broader class of lead field expansions by a simulation study [17]. The simulations used the expression (7) in the form obtained by multiplying both sides by $|\mathbf{j}(\mathbf{r})|$, so that the left hand side appears to have the full transverse two degrees of freedom, as contained in the original form (2). This new form was then analysed by iterating the manipulated form of (7) and the data conditions of equation (1), using as a starting choice $|\mathbf{j}(\mathbf{r})| = constant$. It was found that the linear power dependence of $w(\mathbf{r}, \mathbf{j}(\mathbf{r}))$ on $|\mathbf{j}(\mathbf{r})|$ gave the best results [17].

In a later section we will attempt to discern an underlying principle which might (or might not) single out this simplest form of MFT, using the more general lead field expansions of 'order p', discussed already in [17]. In this more general scheme equation(3) corresponds to a single term expansion with order -1 and the simplest MFT expansion (6) is of order 0. It will be also shown that FOCUSS (when suitably modified to preserve rotational symmetry) uses exactly the same form of lead field 'power' expansion in the source strength, and it closely corresponds to the expansion of order 1.

III. The Initial Argument for Simple MFT

Let us contrast the two choices (2 and 3) and (7) in more detail in the case of a general source distribution $\mathbf{j}$. For the MFT case (7) equation (1) becomes

$$d_m = \sum_n \mathbf{L}_{mn}(|\mathbf{j}|) A_n \tag{8}$$

where

$$\mathbf{L}_{mn}(|\mathbf{j}|) = \int \left(\phi_m(\mathbf{r}_0) \cdot \phi_n(\mathbf{r}_0) \right) w(\mathbf{r}_0) \, |\mathbf{j}(\mathbf{r}_0)| \, d\mathbf{r}_0 \tag{9}$$

The solution to (8) is of form (neglecting problems of poorly defined inversion)

$$\mathbf{A} = \mathbf{L}^{-1}(|\mathbf{j}|)\mathbf{d} \tag{10}$$

in which there is explicit dependence on the length of the sources at each point $\mathbf{r}_0$, $|j(\mathbf{r}_0)|$, of considerably complex non-linearity for a number of sources, arising from the inverse of the matrix L as defined by equation (9). This strength for the source distribution is still undetermined. That unknown quantity at each source point may now be obtained from (7) by the condition that the vector norm of the right hand side of that equation be unity at each source point, or equivalently that the following infinite set of equations be valid:

$$1 = \left| \sum_m \left(\left(\mathbf{L}(|\mathbf{j}|) \right)^{-1} \mathbf{d} \right)_m \phi_m(\mathbf{r}_0) w(\mathbf{r}_0) \right| \tag{11}$$

Mathematical Analysis of Lead Field Expansions

John G. Taylor, Andreas A. Ioannides* and Hans-Wilhem Müller-Gärtner

Abstract—

The solution to the bioelectromagnetic inverse problem is discussed in terms of a generalised lead field expansion, extended to weights depending polynomially on the current strength; the expansion coefficients are obtained from the resulting system of equations which relate the lead field expansion to the data. The framework supports a family of algorithms which include the class of minimum norm solutions and those of weighted minimum norm, including FOCUSS (suitably modified to conform to requirements of rotational invariance). The weighted minimum norm family is discussed in some detail making explicit the dependence (or independence) of the weighting scheme on the modulus of the unknown current density vector; for all but the linear case, and with a single power in the weight, a highly nonlinear system of equations results. These are analysed and their solution reduced to tractable problems for a finite number of degrees of freedom. In the simplest, Magnetic Field Tomography (MFT), case this is shown to possess expected properties for localised distributed sources. A sensitivity analysis supports this conclusion.

Keywords— **Magnetoencephalography, Biomagnetic Inverse Problem, Magnetic Field Tomography**

I. Introduction

For many years cartography of signals and semiology of sources, i.e. equivalent current dipole (ECD) modeling, have dominated the analysis of electrographic (EEG and MEG) data. By the mid 80s the suggestion to use the sensitivity profile of the sensors as a basis for expanding continuous distributions of current had received attention [12] together with Fourier based techniques [8]. The distributed source examples from these early efforts were mainly two-dimensional and they resulted in distributions which were too diffused,failing to highlight enough the known foci of activity in simulations and the expected focal activations in real data. The notable contributions in the next few years include the mathematical summary of Sarvas [26] and simulation comparisons [18]. The work of Clarke [6], [3] has been decisive in establishing a mathematical framework for the treatment of continuous source distributions. The seventh biomagnetism conference in New York marks a transition where algorithms for distributed sources have been described and applied in earnest to the analysis of real data [20], [5], [16], [23], [24]. In the 90s new algorithms have continued to appear and old ones improved (the 'new ones' did not always differ in content from earlier ones). Multiple dipole models have been analysed in depth, while Fourier methods have been revisited [25], [2], [19]. Weighted minimum norm solutions have constantly improved in sophistication in treating the statistical properties of the signal [1], [4], [9], [21]. The introduction of anatomical constraints has been proposed by many authors, and implemented independently and distinctly by others [7], [27], [9].

Of the many approaches now in vogue the evidence favours distributed sources which can also identify sparse solutions, particularly if single trials and spontaneous activity are to be considered for analysis. In this paper we will study the mathematical underpinning of the main weighted minimum norm [12], [13] and MFT [16], [17], [22] algorithms, both singly and in their relationship. The introduction of MFT relied heavily on simulation results and validation studies which provided heuristic support for choosing a lead field expansion which was proportional to the first power of the current. The weight function is chosen by a 'training process' on appropriate computer generated data.

The more recent algorithm FOCUSS [10], [11], is very similar in appearance to MFT, using also an iterative process. However, in its published formulation, it does not assume a weighted lead field expansion for the current sources; we will show, that, when the "city block distance" used in the published version is replaced by the rotationally invariant Euclidean norm, the FOCUSS algorithm is expressible in terms of a generalised MFT lead field expansion. It is also claimed that FOCUSS involves training, which however we distinguish from the one in MFT where a distinct training precedes the processing of the data, so that the algorithm is 'fine tuned' leading to more accurate reconstructions within the source space.

This study is therefore undertaken as a critique of MFT in particular, and distributed sources in general. It will do so in terms of an initial description of the lead field expansions, and of the properties of the lead fields themselves.

II. The Basic Assumption

The effect of the current source distribution on the m^{th} measurement coil may be expressed in terms of the lead field $\phi_m(\mathbf{r})$ for that coil in terms of which a current source distribution $\mathbf{j}(\mathbf{r})$ leads to a measurement value d_m as

$$d_m = \int_{ß} \phi_m(\mathbf{r}) \cdot \mathbf{j_0}(\mathbf{r}) d\mathbf{r} \qquad (1)$$

This work was carried out while the authors worked in the Institute of Medicine at the Research Center Jülich in Germany and it expanded on earlier work by AAI and JGT at the Open University and King's College London in the UK. *Asterisk indicates corresponding author*

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